

FINITE 2-GROUPS OF CLASS 2 WITH SPECIFIC AUTOMORPHISM GROUP

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ABSTRACT. In this paper we classify all finite 2-groups of class 2 for which every automorphism of order 2 leaving the Frattini subgroup elementwise fixed is inner. We prove that every such group G is isomorphic to $Q(n, r) = \langle a, b \mid a^{2^n} = b^{2^r} = 1, a^{2^{n-r}} = [a, b] \rangle$ for some positive integers r, n such that $2 < 2r \leq n$; and every automorphism of $Q(n, r)$ of order 2 leaving the Frattini subgroup elementwise fixed is inner.

1. INTRODUCTION

Let p be a prime number and G be a non-abelian finite p -group. A longstanding conjecture [9] asserts G possesses a non-inner automorphism of order p . By the celebrated result of W. Gaschütz [6], non-inner automorphisms of G of p -power order exist. Deaconescu and Silberberg [5] proved that a finite p -group G satisfying the condition $C_G(Z(\Phi(G))) \neq \Phi(G)$ has a noninner automorphism of order p leaving the Frattini subgroup $\Phi(G)$ elementwise fixed. Liebeck [8] has shown the same result when G is nilpotent of class 2 and $p > 2$. He also showed that the latter is not valid for $p = 2$ by giving an example of a finite 2-group of class 2 and order 128 in which every automorphism of order 2 fixing $\Phi(G)$ elementwise is inner. The first author [2] exhibited another example of a finite 2-group of order 64 and class 2 in which every automorphism of order 2 fixing $\Phi(G)$ is inner. In [1] it is shown that every 2-group of class 2 has a non-inner automorphism of order 2 fixing $\Omega_1(Z(G))$ or $\Phi(G)$ elementwise (actually the same proof in [1] shows that such an automorphism fixes $Z(G)$ or $\Phi(G)$ elementwise).

In this paper we classify all finite 2-groups of class 2 for which every automorphism of order 2 leaving the Frattini subgroup elementwise fixed is inner.

Theorem 1.1. If G is a finite 2-group of class 2 for which every automorphism of order 2 leaving the Frattini subgroup elementwise fixed is inner, then G is isomorphic to $\langle a, b \mid a^{2^n} = b^{2^r} = 1, a^{2^{n-r}} = [a, b] \rangle$, $2 < 2r \leq n$.

Throughout the paper p denotes a prime number. For a group G , $Z(G)$, G' , $\Phi(G)$ and $d(G)$ denote the center, the derived subgroup, the Frattini subgroup and the minimum number of generators of G , respectively. The inner automorphism induced by $x \in G$ is denoted by θ_x . If G and A are groups, $\text{Hom}(G, A)$ denotes the set of all homomorphisms from G to A . If A is an abelian group $\text{Hom}(G, A)$ with pointwise multiplication forms an abelian group.

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2. REDUCTION TO 2-GENERATOR GROUPS

Let G be a finite 2-group of class 2 such that every automorphism of order 2 leaving the Frattini subgroup of G elementwise fixed is inner. Call this condition (\star) .

Remark 2.1. ([2], Remark 2.4) Let G be a finite p -group of class 2. If G has no non-inner automorphism of order p leaving $\Phi(G)$ elementwise fixed, then $Z(G)$ must be cyclic.

Remark 2.2. If $\Omega_1(Z(G)) \leq \Phi(G)$ and $f \in \text{Hom}(\frac{G}{\Phi(G)}, \Omega_1(Z(G)))$, then it is well-known that the mapping ϕ_f defined by $x^{\phi_f} = x\bar{x}^f$ where $\bar{x} = x\Phi(G)$ for any $x \in G$, is an automorphism of order 2 that fixes $\Phi(G)$ elementwise. If ϕ_f is inner, then $\phi_f = \theta_x$ for some $x \in G$. Since $\theta_x|_{\Phi(G)} = \text{id}$ one has $x \in C_G(\Phi(G))$. If G satisfies the condition (\star) , then it follows from [5] that $C_G(Z(\Phi(G))) = \Phi(G)$ and so $C_G(\Phi(G)) = Z(\Phi(G))$. Thus $x \in Z(\Phi(G))$. Now (\star) implies that

$$d(G)d(Z(G)) = d(\text{Hom}(\frac{G}{\Phi(G)}, \Omega_1(Z(G)))) \leq d(\frac{Z(\Phi(G))}{Z(G)}) \leq d(Z(\Phi(G))),$$

therefore we have $d(G) \leq d(Z(\Phi(G)))$ for any group G satisfying the condition (\star) .

Lemma 2.3. Let G be a finite group and $A \trianglelefteq G$, $B \leq G$ and $G = AB$. Then $\alpha \in \text{Aut}(A)$ and $\beta \in \text{Aut}(B)$ have a common extension to G if and only if α and β agree on $A \cap B$ and $[a, b]^\alpha = [a^\alpha, b^\beta]$ for all $a \in A, b \in B$.

Proof. It is straightforward. \square

Theorem 2.4. Let G be a finite 2-group of class 2 and $\frac{G}{\Phi(G)} = \langle \bar{x}_1 \rangle \times \cdots \times \langle \bar{x}_d \rangle$ for some elements $x_1, \dots, x_d \in G$ where $\bar{x} = x\Phi(G)$ for any $x \in G$. If $b_1, \dots, b_d \in \Omega_1(Z(\Phi(G)))$ such that $[x_i, b_i] = 1$ and $[x_i, b_j] = [x_j, b_i]$ for $1 \leq i < j \leq d$, then the mapping x_i to $x_i b_i$, for $1 \leq i \leq d$, can be extended to an automorphism of G of order 2.

Proof. For $1 \leq j \leq d$, let $H_j = \langle x_1, \dots, x_j, \Phi(G) \rangle$. We prove by induction on j that $\left| \begin{smallmatrix} x_i \mapsto x_i b_i \\ 1 \leq i \leq j \end{smallmatrix} \right|$ determines an automorphisms of order 2 on H_j which fixes the $\Phi(G)$ element-wise. For $j = 1$, if $h \in H_1$, then $h = x_1^i m$, for some non-negative integer i and $m \in \Phi(G)$. Naturally define $h^\alpha = (x_1 b_1)^i m$. Now it is easy to see that α is the desired automorphism on H_1 . Now suppose that α is the automorphism on H_j determined by $\left| \begin{smallmatrix} x_i \mapsto x_i b_i \\ 1 \leq i \leq j \end{smallmatrix} \right|$ and β is the automorphism on $\langle x_{j+1}, \Phi(G) \rangle$ defined by $x_{j+1} \rightarrow x_{j+1} b_{j+1}$. Now by Lemma 2.3., α and β have a common extension to an automorphism on H_{j+1} . \square

Theorem 2.5. Let G be a finite 2-group of class 2 satisfying (\star) . Then $d(G) \leq 3$.

Proof. Let $\frac{G}{\Phi(G)} = \langle \bar{x}_1 \rangle \times \cdots \times \langle \bar{x}_{d(G)} \rangle$ for some elements $x_1, \dots, x_{d(G)} \in G$. Define

$$\varphi : \underbrace{\Omega_1(Z(\Phi(G))) \times \cdots \times \Omega_1(Z(\Phi(G)))}_{d(G)-\text{times}} \mapsto \underbrace{\Omega_1(Z(G)) \times \cdots \times \Omega_1(Z(G))}_{\binom{d(G)+1}{2}-\text{times}}$$

by $(b_1, \dots, b_{d(G)})^\varphi = ([x_1, b_1], \dots, [x_{d(G)}, b_{d(G)}], \dots, [x_i, b_j][b_i, x_j], \dots)$. It is easy to see that φ is a homomorphism and if $(b_1, \dots, b_{d(G)}) \in \ker(\varphi)$, then by Theorem

2.4. the mapping $\left| \begin{smallmatrix} x_i \mapsto x_i b_i \\ 1 \leq i \leq j \end{smallmatrix} \right|$ determines an automorphism of order 2 leaving $\Phi(G)$ elementwise fixed. Since the domain of φ is elementary abelian, then

$$d(\ker(\varphi)) \geq d(G)d(Z(\Phi(G))) - \binom{d(G)+1}{2}.$$

By main result of [5] we may assume that $C_G(Z(\Phi(G))) = \Phi(G)$. Therefore it follows from Remark 2.2. that $d(\ker(\varphi)) \geq \binom{d(G)}{2}$. Now the condition (\star) implies that $d(G) \geq d(\ker(\varphi)) \geq \binom{d(G)}{2}$ and so $d(G) \leq 3$. \square

Lemma 2.6. Let G be a finite 2-group of class 2 satisfying (\star) . Then $d(G) \neq 3$.

Proof. Suppose that $G = \langle x_1, x_2, x_3 \rangle$. We may assume that $G' = \langle [x_1, x_2] \rangle$, since $Z(G)$ is cyclic by Remark 2.1. Therefore $[x_1, x_2]^i = [x_1, x_3]$, $[x_1, x_2]^j = [x_2, x_3]$, for some integers i, j . Then $[x_1, x_2^{-i} x_3] = 1$ and $[x_1^j x_3, x_2] = 1$. Hence $[x_2, x_1^j x_2^{-i} x_3] = 1$ and $[x_1, x_1^j x_2^{-i} x_3] = 1$ and so $x_1^j x_2^{-i} x_3 \in Z(G) \leq \Phi(G)$ (note that by [5], $Z(G) \leq \Phi(G)$). Therefore $G = \langle x_1, x_2 \rangle$, a contradiction. This completes the proof. \square

3. PROOF OF THEOREM 1.1.

Let G be a finite 2-group of class 2 satisfying condition (\star) . By Section 2, we may assume that $d(G) = 2$ and $Z(G)$ is cyclic. The proof of Theorem 1.1. is as follows:

Y. K. Leong [7] has given a complete classification of 2-generator 2-groups of class 2 with cyclic center. The classification is as follows:

- (1) $Q(n, r) = \langle a, b \mid a^{2^n} = b^{2^r} = 1, a^{2^{n-r}} = [a, b] \rangle$ and $2r \leq n$.
- (2) $Q(n, r) = \langle a, b \mid a^{2^n} = b^{2^r} = 1, a^{2^r} = [a, b]^{2^{2^r-n}}, [[a, b], a] = [[a, b], b] = 1 \rangle$ and $r \leq n < 2r$.
- (3) $R(n) = \langle a, b \mid a^{2^{n+1}} = b^{2^{n+1}} = 1, a^{2^n} = [a, b]^{2^{2^n-1}} = b^{2^n}, [[a, b], a] = [[a, b], b] = 1 \rangle$ and $n \geq 1$.

Thus, to complete the proof of Theorem 1.1. it is enough to check which of these groups satisfy (\star) . To this end, first note that if a 2-generator p -group G of class 2 has a presentation

$$\langle a, b \mid r_\ell(a, b), 1 \leq \ell \leq m \rangle,$$

then every element $g \in G$ has the form $g = a^i b^j [a, b]^k$ for some integers i, j and k . Moreover, if $G = \langle a^i b^j [a, b]^k, a^{i'} b^{j'} [a, b]^{k'} \rangle$ and $r_\ell(a^i b^j [a, b]^k, a^{i'} b^{j'} [a, b]^{k'}) = 1$, for all $\ell \in \{1, \dots, m\}$, then by Von Dyck's theorem the mapping $\left| \begin{smallmatrix} a \mapsto a^i b^j [a, b]^k \\ b \mapsto a^{i'} b^{j'} [a, b]^{k'} \end{smallmatrix} \right|$ determines an automorphism of G . Next, we apply the following result to verify whether a given automorphism of G is inner.

Remark 3.1. ([4], Part (ii) of Lemma 1) Suppose that G is a finite 2-generator 2-group of class 2, such that $G' = \langle a \rangle$. Then $\alpha \in \text{Aut}(G)$ is inner if and only if $[G, \alpha] \leq G'$.

We now start to check groups $Q(n, r)$, $R(n)$.

- (1) Let $G = Q(n, r) = \langle a, b \mid a^{2^n} = b^{2^r} = 1, a^{2^{n-r}} = [a, b] \rangle$ and $2r \leq n$;
 - (i) If $G = Q(2, 1)$, then $G \simeq D_8$ and does not satisfy (\star) because it is easy to see that the following map α can be extended to a noninner automorphism

of order 2 leaving the Frattini subgroup of G elementwise fixed.

$$\alpha : \begin{cases} a \mapsto a^3 \\ b \mapsto ab \end{cases}$$

- (ii) If $G = Q(n, 1)$, then G does not satisfy (\star) because G has exactly two non-inner automorphisms of order 2 which act trivially on the Frattini subgroup as follow:

$$\alpha : \begin{cases} a \mapsto a^{1+2^{n-2}+m2^{n-1}}b \\ b \mapsto a^{2^{n-1}}b \end{cases}, \quad m \in \{0, 1\}$$

- (iii) If $n > 2$, then every automorphism of G which is leaving the Frattini subgroup elementwise fixed, maps the generators a and b as either the map α_1 or α_2 :

$$\alpha_1 : \begin{cases} a \mapsto a^{1+m2^{n-1}} \\ b \mapsto a^{s2^{n-1}}b \end{cases}, \quad \alpha_2 : \begin{cases} a \mapsto a^{1+2^{n-2}+m2^{n-1}}b^{2^{r-1}} \\ b \mapsto a^{s2^{n-1}}b \end{cases}, \quad m, s \in \{0, 1\}$$

Now Remark 3.1. implies that $\alpha_1 \in \text{Inn}(G)$ and $\alpha_2 \notin \text{Inn}(G)$. On the other hand $|\alpha_2| \neq 2$. Therefore G satisfies the condition (\star) .

- (2) Let $G = Q(n, r) = \langle a, b \mid a^{2^n} = b^{2^r} = 1, a^{2^r} = [a, b]^{2^{2r-n}}, [[a, b], a] = [[a, b], b] = 1 \rangle$ where $r \leq n < 2r$.

- (i) If $n = r = 1$, let $\alpha : \begin{cases} a \mapsto b \\ b \mapsto a \end{cases}$,
- (ii) If $n = r > 1$ let $\alpha : \begin{cases} a \mapsto a^{2^{r-1}+1} \\ b \mapsto b^{2^{r-1}+1} \end{cases}$,
- (iii) If $2r > n \geq r + 1$, $r > 1$, let $\alpha : \begin{cases} a \mapsto a^{2^{n-1}-2^{r-1}+1}[a, b]^{2^{2r-n-1}} \\ b \mapsto a^{2^{n-1}}b^{2^{r-1}+1} \end{cases}$,

Then it can be verified that in each case α determines a non-inner automorphism of order 2 leaving the Frattini subgroup of G elementwise fixed.

- (3) Let $G = R(n) = \langle a, b \mid a^{2^{n+1}} = b^{2^{n+1}} = 1, a^{2^n} = [a, b]^{2^{n-1}} = b^{2^n}, [[a, b], a] = [[a, b], b] = 1 \rangle$, where $n \geq 1$;

- (i) If $n = 1$, let $\alpha : \begin{cases} a \mapsto ab \\ b \mapsto b^3 \end{cases}$,
- (ii) If $n \geq 2$, let $\alpha : \begin{cases} a \mapsto a^{2^n+2^{n-1}+1}[a, b]^{2^{n-2}} \\ b \mapsto b^{2^n+2^{n-1}+1}[a, b]^{2^{n-2}} \end{cases}$.

Then it can be verified that in each case α is a non-inner automorphism of order 2 leaving the Frattini subgroup of G elementwise fixed.

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